Guts Round Scoring System

The format for the Guts Round is different from the other tests. The Guts Round lasts 75 minutes, and is divided into 9 sets of questions. You will start with Set 1, and will receive Set $k+1$ after submitting Set k. You can take as much time as you wish for each set, but you cannot go back to a previous set after submitting it.

Sets 1 through 8 consist of short answer questions, similar to the earlier tests. Either your answer is correct or incorrect, and points are given accordingly.

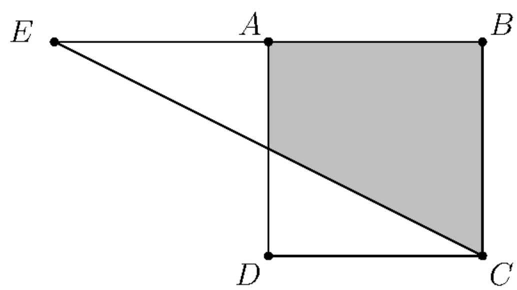
Set 9 (the last set) consists of estimation questions. Your score is based on how close your answer is to the correct answer. More details about scoring are given in Set 9.

Each question within the set is equally weighted, but each set is weighted differently.

- Set 1: 10 points per question
- Set 2: 11 points per question
- Set 3: 12 points per question
- Set 4: 13 points per question
- Set 5: 14 points per question
- Set 6: 16 points per question
- Set 7: 18 points per question
- Set 8: 21 points per question
- Set 9: 25 points per question

Set 1

1. Given a square $ABCD$ of side length 6, the point E is drawn on the line AB such that the distance EA is less than EB and the triangle $\triangle BCE$ has the same area as $ABCD$. Compute the shaded area.



2. Jerry has red blocks, yellow blocks, and blue blocks. He builds a tower 5 blocks high, without any 2 blocks of the same color touching each other. Also, if the tower is flipped upside-down, it still looks the same. Compute the number of ways Jerry could have built this tower.

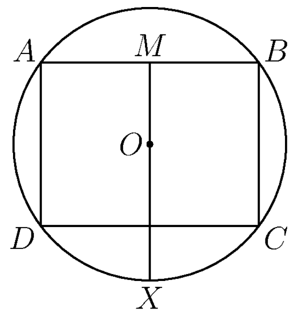
3. Compute the second smallest positive whole number that has exactly 6 positive whole number divisors (including itself).

Set 2

4. A grasshopper is traveling on the coordinate plane, starting at the origin $(0, 0)$. Each hop, the grasshopper chooses to move 1 unit up, down, left, or right with equal probability. The grasshopper hops 4 times and stops at point P. Compute the probability that it is possible to return to the origin from P in at most 3 hops.

5. Two parabolas, $y = ax^2 + bx + c$ and $y = -ax^2 - bx - c$, intersect at $x = 2$ and $x = -2$. If the y-intercepts of the two parabolas are exactly 2 units apart from each other, compute $|a + b + c|$.

6. Let rectangle $ABCD$ have side lengths $AB = 8, BC = 6$. Let $ABCD$ be inscribed in a circle with center O, as shown in the diagram. Let M be the midpoint of side $\overline{AB}$, and let X be the intersection of ray $\overrightarrow{MO}$ with the circle. Compute the length AX.



Set 3

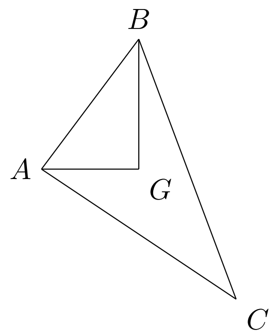
7. For an integer $n > 0$, let $p(n)$ be the product of the digits of n. Compute the sum of all integers n such that $n - p(n) = 52$.

8. Circle ω_1 is centered at O_1 with radius 3, and circle ω_2 is centered at O_2 with radius 2. Line ℓ is tangent to ω_1 and ω_2 at X, Z, respectively, and intersects segment $\overline{O_1O_2}$ at Y. The circle through O_1, X, Y has center O_3, and the circle through O_2, Y, Z has center O_4. Given that $O_1O_2 = 13$, find O_3O_4.

9. For positive integers a and b, consider the curve $x^a + y^b = 1$ over real numbers x, y and let $S(a, b)$ be the sum of the number of x-intercepts and y-intercepts of this curve. Compute $\sum_{a=1}^{10} \sum_{b=1}^{5} S(a, b)$.

Set 4

10. Let $\triangle ABC$ be a triangle with G as its centroid, which is the intersection of the three medians of the triangle, as shown in the diagram. If $\overline{GA} \perp \overline{GB}$ and $AB = 7$, compute $AC^2 + BC^2$.



11. Compute the sum of all positive integers n for which there exists a real number x satisfying

$$\left(x + \frac{n}{x}\right)^n = 2^{20}.$$

12. Call an n-digit integer with distinct digits *mountainous* if, for some integer $1 \leq k \leq n$, the first k digits are in strictly ascending order and the following $n - k$ digits are in strictly descending order. How many 5-digit mountainous integers with distinct digits are there?

Set 5

13. Consider the set of triangles with side lengths $1 \leq x \leq y \leq z$ such that x, y, and z are the solutions to the equation $t^3 - at^2 + bt = 12$ for some real numbers a and b. Compute the smallest real number N such that $N > ab$ for any choice of x, y, and z.

14. Right triangle $\triangle ABC$ with $\angle A = 30^\circ$ and $\angle B = 90^\circ$ is inscribed in a circle ω_1 with radius 4. Circle ω_2 is drawn to be the largest circle outside of $\triangle ABC$ that is tangent to both $\overline{BC}$ and ω_1, and circles ω_3 and ω_4 are drawn this same way for sides $\overline{AC}$ and $\overline{AB}$, respectively. Suppose that the intersection points of these smaller circles with the bigger circle are noted as points D, E, and F. Compute the area of triangle $\triangle DEF$.

15. Given a positive integer k, let $s(k)$ denote the sum of the digits of k. Let $a_1, a_2, a_3, \ldots$ denote the strictly increasing sequence of all positive integers n such that $s(7n+1) = 7s(n) + 1$. Compute a_{2023}.

Set 6

16. Sabine rolls a fair 14-sided die numbered 1 to 14 and gets a value of x. She then draws x cards uniformly at random (without replacement) from a deck of 14 cards, each of which labeled a different integer from 1 to 14. She finally sums up the value of her die roll and the value on each card she drew to get a score of S. Let A be the set of all obtainable scores. Compute the probability that S is greater than or equal to the median of A.

17. Let N be the smallest positive integer divisble by $10^{2023} - 1$ that only has the digits 4 and 8 in decimal form (these digits may be repeated). Compute the sum of the digits of $\frac{N}{10^{2023}-1}$.

18. Consider the sequence $b_1, b_2, b_3, \ldots$ of real numbers defined by $b_1 = \frac{3+\sqrt{3}}{6}$, $b_2 = 1$, and for $n \geq 3$,

$$b_n = \frac{1 - b_{n-1} - b_{n-2}}{2b_{n-1}b_{n-2} - b_{n-1} - b_{n-2}}.$$

Compute b_{2023}.

Set 7

19. Let N_{21} be the answer to question 21. Suppose a jar has $3N_{21}$ colored balls in it: N_{21} red, N_{21} green, and N_{21} blue balls. Jonathan takes one ball at a time out of the jar uniformly at random without replacement until all the balls left in the jar are the same color. Compute the expected number of balls left in the jar after all balls are the same color.

20. Let N_{19} be the answer to question 19. For every non-negative integer k, define

$$f_k(x) = x(x-1) + (x+1)(x-2) + \cdots + (x+k)(x-k-1),$$

and let r_k and s_k be the two roots of $f_k(x)$. Compute the smallest positive integer m such that $|r_m - s_m| > 10N_{19}$.

21. Let N_{20} be the answer to question 20. In isosceles trapezoid $ABCD$ (where $\overline{BC}$ and $\overline{AD}$ are parallel to each other), the angle bisectors of A and D intersect at F, and the angle bisectors of points B and C intersect at H. Let $\overline{BH}$ and $\overline{AF}$ intersect at E, and let $\overline{CH}$ and $\overline{DF}$ intersect at G. If $CG = 3$, $AE = 15$, and $EG = N_{20}$, compute the area of the quadrilateral formed by the four tangency points of the largest circle that can fit inside quadrilateral $EFGH$.

Set 8

22. Let $d_n(x)$ be the nth decimal digit (after the decimal point) of x. For example, $d_3(\pi) = 1$ because $\pi = 3.14\underline{1}5\ldots$. For a positive integer k, let $f(k) = p_k^4$, where p_k is the kth prime number. Compute the value of $\sum_{i=1}^{2023} d_{f(i)}\left(\frac{1}{1275}\right)$.

23. A robot initially at position 0 along a number line has a *movement function* $f(u, v)$. It rolls a fair 26-sided die repeatedly, with the kth roll having value r_k. For $k \geq 2$, if $r_k > r_{k-1}$, it moves $f(r_k, r_{k-1})$ units in the positive direction. If $r_k < r_{k-1}$, it moves $f(r_k, r_{k-1})$ units in the negative direction. If $r_k = r_{k-1}$, all movement and die-rolling stops and the robot remains at its final position x. If $f(u, v) = (u^2 - v^2)^2 + (u - 1)(v + 1)$, compute the expected value of x.

24. Define the sequence $s_0, s_1, s_2, \ldots$ by $s_0 = 0$ and $s_n = 3s_{n-1} + 2$ for $n \geq 1$. The monic polynomial $f(x)$ defined as

$$f(x) = \frac{1}{s_{2023}} \sum_{k=0}^{32} s_{2023+k} x^{32-k}$$

can be factored uniquely (up to permutation) as the product of 16 monic quadratic polynomials $p_1, p_2, \cdots, p_{16}$ with real coefficients, where $p_i(x) = x^2 + a_i x + b_i$ for $1 \leq i \leq 16$. Compute the integer N that minimizes $\left|N - \sum_{k=1}^{16}(a_k + b_k)\right|$.

Set 9

25. Let triangle $\triangle ABC$ have side lengths $AB = 6$, $BC = 8$, and $CA = 10$. Let S_1 be the largest square fitting inside of $\triangle ABC$ (sharing points on edges is allowed). Then, for $i \geq 2$, let S_i be the largest square that fits inside of $\triangle ABC$ while remaining outside of all other squares $S_1, \cdots, S_{i-1}$ (with ties broken arbitrarily). For all $i \geq 1$, let m_i be the side length of S_i and let S be the set of all m_i. Let x be the 2023rd largest value in S. Compute $\log_2\left(\frac{1}{x}\right)$.

    Submit your answer as a decimal E to at most 3 decimal places. If the correct answer is A, your score for this question will be $\max(0, 25 - 2|A - E|)$, rounded to the nearest integer.

26. For positive integers i and N, let $k_{N,i}$ be the ith smallest positive integer such that the polynomial $\frac{x^2}{2023} + \frac{Nx}{7} - k_{N,i}$ has integer roots. Compute the minimum positive integer N satisfying the condition $\frac{k_{N,2023}}{k_{N,1000}} < 3$. Submit your answer as a positive integer E. If the correct answer is A, your score for this question will be $\max\left(0, 25\min\left(\frac{A}{E}, \frac{E}{A}\right)^{\frac{3}{2}}\right)$, rounded to the nearest integer.

27. Let ω be a circle with positive integer radius r. Suppose that it is possible to draw isosceles triangle with integer side lengths inscribed in ω. Compute the number of possible values of r where $1 \leq r \leq 2023^2$. Submit your answer as a positive integer E. If the correct answer is A, your score for this question will be $\max\left(0, 25\left(3 - 2\max\left(\frac{A}{E}, \frac{E}{A}\right)\right)\right)$, rounded to the nearest integer.