Time Limit: 60 mins.
Maximum Score: 100 points.
Instructions:

1. You may use the result of a previous problem that you have not solved in order to solve a later problem without losing credit on the later problem.

2. Unless otherwise stated, all answers must be justified by proofs.

3. FORMAT FOR SOLUTIONS: On each sheet of paper, draw 1 inch margins. Put your solutions in the center. Write your team ID and the problem number in the upper- right hand corner. It is okay to use both sides of the paper for the same problem, but you must use a separate sheet of paper for every problem.

4. If you use multiple sheets for a problem, clearly label them 1/2, 2/2, etc.

5. Write your team ID on every page that you turn in.

6. Points will be deducted for answers that have many incorrect statements or an excessive amout of irrelevant information. The minimum that you can receive for any question is zero points. Use your best judgment here. The goal is not to penalize you for making mistakes.

7. If you make partial progress on a problem, make it clear what you have done and what you need to show.

Comments: The mathematics that you are about to see plays an essential role in quantum mechanics, an area of physics that studies very small objects. Furthermore, the material in this power round just scratches the surface of a vast and interesting area of math and physics. If you are interested in learning more, some key words to look into are: lie algebras, representation theory, operators in quantum mechanics.

Good luck and I hope that you enjoy getting exposed to some new ideas! -Alex

1 Introduction (0 pts)

The Algebra of Noncommutative Operators: In this power round, we will consider the algebra of noncommutative operators. We will define operators as *any objects* that satisfy the properties below. We will use bold-faced, upper case letters, such as $\mathbf{A}$, to denote operators and lower case letters to denote complex numbers. We can add operators, multiply two operators, and multiply operators by numbers. Almost all of the properties that we take for granted for real numbers hold for operators. The only exception is that $\mathbf{AB} \neq \mathbf{BA}$ where $\mathbf{A}, \mathbf{B}$ are operators. Also you cannot add a number and an operator. In particular, the following properties hold:

Suppose that $a, b, \ldots$ are complex numbers and $\mathbf{A}, \mathbf{B}, \ldots$ are operators. You may use the following properties without explicitly stating them:

(a) $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$

(b) $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$.

(c) $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$.

(d) $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$, $(\mathbf{A} + \mathbf{B})\mathbf{C} = \mathbf{AC} + \mathbf{BC}$.

(e) $(a + b)\mathbf{C} = a\mathbf{C} + b\mathbf{C}$.

(f) $a(\mathbf{B} + \mathbf{C}) = a\mathbf{B} + a\mathbf{C}$

(g) $a(b\mathbf{C}) = (ab)\mathbf{C}$.

(h) $\mathbf{A}(b\mathbf{C}) = b(\mathbf{AC}) = (b\mathbf{A})\mathbf{C}$.

(i) There is a zero operator, often written as $\mathbf{0}$ such that $\mathbf{0} + \mathbf{A} = \mathbf{A}$ for all operators $\mathbf{A}$.

(j) $0\mathbf{A} = \mathbf{0A} = \mathbf{A0} = \mathbf{0}$ (where $\mathbf{0}$ is the zero operator and 0 is the complex number zero).

(k) Given a operator, $\mathbf{A}$, there exists another operator, denoted by $-\mathbf{A}$ such that $\mathbf{A} + (-\mathbf{A}) = \mathbf{0}$.

(l) There exists a multiplicative identity for operators called the identity operator, often written as $\mathbf{I}$. The identity operator is such that $\mathbf{IA} = \mathbf{AI} = \mathbf{A}$ for all operators $\mathbf{A}$.

2 Manipulating Commutators (25 pts)

Commutator: As discussed above, it is not generally true that $\mathbf{AB} = \mathbf{BA}$ for operators. Therefore, it can be useful to consider the quantity, $\mathbf{AB} - \mathbf{BA}$, which is not generally zero. This quantity is called the commutator of $\mathbf{A}$ and $\mathbf{B}$ and is written as follows

$$[\mathbf{A}, \mathbf{B}] = \mathbf{AB} - \mathbf{BA} \tag{1}$$

P1 (a) **(2 pts)** Prove that $[\mathbf{A}, \mathbf{A}] = \mathbf{0}$ and $[\mathbf{A}, \mathbf{B}] = -[\mathbf{B}, \mathbf{A}]$.

(b) **(2 pts)** Prove the following two properties of commutators

$$[\mathbf{A}, \mathbf{BC}] = \mathbf{B}[\mathbf{A}, \mathbf{C}] + [\mathbf{A}, \mathbf{B}]\mathbf{C} \qquad [\mathbf{A}, b\mathbf{B} + c\mathbf{C}] = b[\mathbf{A}, \mathbf{B}] + c[\mathbf{A}, \mathbf{C}] \tag{2}$$

where $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are operators and a, b are numbers.

(c) **(2 pts)** Prove the Jacobi Identity:

$$[\mathbf{A}, [\mathbf{B}, \mathbf{C}]] + [\mathbf{B}, [\mathbf{C}, \mathbf{A}]] + [\mathbf{C}, [\mathbf{A}, \mathbf{B}]] = \mathbf{0} \tag{3}$$

P2 **(3 pts)** Suppose that $[\mathbf{A}, \mathbf{B}] = \mathbf{0}$. If n is a positive integer, prove that $[\mathbf{A}^n, \mathbf{B}] = \mathbf{0}$.

For your later reference, it is also true that

$$[\mathbf{AB}, \mathbf{C}] = \mathbf{A}[\mathbf{B}, \mathbf{C}] + [\mathbf{A}, \mathbf{C}]\mathbf{B} \qquad [a\mathbf{A} + b\mathbf{B}, \mathbf{C}] = a[\mathbf{A}, \mathbf{C}] + b[\mathbf{B}, \mathbf{C}] \tag{4}$$

Angular Momentum Algebra: An ordered triple of three operators, $(\mathbf{J}_1, \mathbf{J}_2, \mathbf{J}_3) = \vec{\mathbf{J}}$, are said to form an angular momenum algebra if

$$[\mathbf{J}_1, \mathbf{J}_2] = i\mathbf{J}_3 \qquad [\mathbf{J}_2, \mathbf{J}_3] = i\mathbf{J}_1 \qquad [\mathbf{J}_3, \mathbf{J}_1] = i\mathbf{J}_2 \tag{5}$$

Note $i = \sqrt{-1}$ is the imaginary unit.

Later in the power round, it will be useful to write the first condition as $[\mathbf{J}_i, \mathbf{J}_j] = i\sum_{k=1}^{3} \epsilon_{i,j,k}\mathbf{J}_k$ for $i = 1, 2, 3$ and $j = 1, 2, 3$ where $\epsilon_{i,j,k}$ is the the Levi-Civita symbol is defined as follows: $\epsilon_{1,2,3} = \epsilon_{2,3,1} = \epsilon_{3,1,2} = 1$ and $\epsilon_{3,2,1} = \epsilon_{2,1,3} = \epsilon_{1,3,2} = -1$. If any of the i, j, k are equal, then $\epsilon_{i,j,k} = 0$.

P3 (a) **(3 pts)** Define $\mathbf{J}_+ = \mathbf{J}_1 + i\mathbf{J}_2$ and $\mathbf{J}_- = \mathbf{J}_1 - i\mathbf{J}_2$. Prove that

$$[\mathbf{J}_3, \mathbf{J}_+] = \mathbf{J}_+ \qquad [\mathbf{J}_3, \mathbf{J}_-] = -\mathbf{J}_- \qquad [\mathbf{J}_+, \mathbf{J}_-] = 2\mathbf{J}_3 \tag{6}$$

(b) **(3 pts)** Define $\mathbf{J}^2 = (\mathbf{J}_1)^2 + (\mathbf{J}_2)^2 + (\mathbf{J}_3)^2$. Prove that $[\mathbf{J}^2, \mathbf{J}_i] = 0$ for $i = 1, 2, 3$.

P4 **(4 pts)** Suppose that

$$\mathbf{J}^2\mathbf{A} = \alpha\mathbf{A} \qquad \mathbf{J}_3\mathbf{A} = \beta\mathbf{A} \tag{7}$$

where $\mathbf{A}$ is a operator and α, β are complex numbers. Prove that

$$\mathbf{J}^2(\mathbf{J}_\pm\mathbf{A}) = \alpha(\mathbf{J}_\pm\mathbf{A}) \qquad \mathbf{J}_3(\mathbf{J}_\pm\mathbf{A}) = (\beta \pm 1)(\mathbf{J}_\pm\mathbf{A}) \tag{8}$$

where you take all $+$'s or all $-$'s.

P5 **(6 pts)** Suppose that we have an operator, $\mathbf{A}$ such that $[\mathbf{J}_3, \mathbf{A}] = k\mathbf{A}$ where k is a non-negative integer. For the sake of notation, write $\mathbf{T}_k = \mathbf{A}$, and define $\mathbf{T}_{q-1} = [\mathbf{J}_-, \mathbf{T}_q]$ for $q = k, k-1, k-2, ...$ Prove that $[\mathbf{J}_3, \mathbf{T}_q] = q\mathbf{T}_q$

3 Counting with the Lie Product Formula (30 pts)

Operator Exponentials: Just as we can take a real number x, and then form a new real number, e^x, we can define the exponential of an operator using a similar formula:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = \sum_{k=0}^{\infty} \frac{x^k}{k!} \;\rightarrow\; \exp(\mathbf{A}) \equiv \mathbf{I} + \mathbf{A} + \frac{\mathbf{A}^2}{2!} + \frac{\mathbf{A}^3}{3!} + ... = \sum_{k=0}^{\infty} \frac{\mathbf{A}^k}{k!} \tag{9}$$

where $\mathbf{A}^0 = \mathbf{I}$ is the identity operator as defined in the introduction.

We know that for real numbers, $e^{x+y} = e^x e^y$. Suppose that instead of real numbers, we have operators, $\mathbf{X}$ and $\mathbf{Y}$. A natural question to ask is: what can we say about $\exp(\mathbf{X}+\mathbf{Y})$? While $\exp(\mathbf{X}+\mathbf{Y}) \neq \exp(\mathbf{X})\exp(\mathbf{Y})$ generally, the Lie product formula gives a way to represent $\exp(\mathbf{X}+\mathbf{Y})$ in terms of products of $\exp\left(\frac{\mathbf{X}}{n}\right)$ and $\exp\left(\frac{\mathbf{Y}}{n}\right)$:

$$\exp(\mathbf{X} + \mathbf{Y}) = \lim_{n\to\infty} \mathbf{M}_n \qquad \text{where} \qquad \mathbf{M}_n = \left(\exp\left(\frac{\mathbf{X}}{n}\right)\exp\left(\frac{\mathbf{Y}}{n}\right)\right)^n \tag{10}$$

If you are not familiar with limits, the intuitive idea is that if we take n to be very large, then the right side becomes the left side. We will not prove the lie product formula, but we will see that there are many counting problems hidden in the polynomial expansion of both sides. Let us consider the example of $n = 2$.

In order to investigate this operator, we need to go back to the definition of the operator exponential and substitute it in to the previous equation to get:

$$\mathbf{M}_2 = \exp\left(\frac{\mathbf{X}}{2}\right)\exp\left(\frac{\mathbf{Y}}{2}\right)\exp\left(\frac{\mathbf{X}}{2}\right)\exp\left(\frac{\mathbf{Y}}{2}\right) \tag{11}$$

$$= \left(\mathbf{I} + \frac{\mathbf{X}}{2} + \frac{1}{2!}\left(\frac{\mathbf{X}}{2}\right)^2 + \dots\right)\left(\mathbf{I} + \frac{\mathbf{Y}}{2} + \frac{1}{2!}\left(\frac{\mathbf{Y}}{2}\right)^2 + \dots\right)\left(\mathbf{I} + \frac{\mathbf{X}}{2} + \frac{1}{2!}\left(\frac{\mathbf{X}}{2}\right)^2 + \dots\right)\left(\mathbf{I} + \frac{\mathbf{Y}}{2} + \frac{1}{2!}\left(\frac{\mathbf{Y}}{2}\right)^2 + \dots\right) \tag{12}$$

Recalling that $\mathbf{XY} \neq \mathbf{YX}$, we can expand out the above expression to get a polynomial function[1] of $\mathbf{X}$ and $\mathbf{Y}$.

P6 **(4 pts)** Find a, b, c, d, e, f, g if after expanding the expression for $\mathbf{M}_2$, we get:

$$\mathbf{M}_2 = a\mathbf{I} + b\mathbf{X} + c\mathbf{Y} + d\mathbf{X}^2 + e\mathbf{XY} + f\mathbf{YX} + g\mathbf{Y}^2 + \dots \tag{13}$$

(No proof necessary).

We call this finding the expansion of $\mathbf{M}_2$ to second order because we are finding the coefficients of all terms with degree less than equal to degree 2.

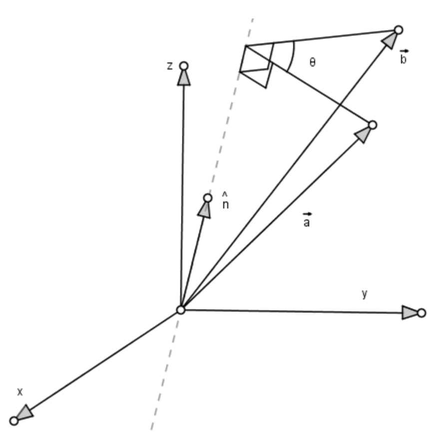
P7 **(7 pts)** Suppose that n is a positive integer. Find, with proof, the expansions of $\mathbf{M}_n$ and $\exp(\mathbf{X}+\mathbf{Y})$ to second order and show that their difference goes to zero if we let n go to infinity.

P8 **(7 pts)** Find, with proof, the coefficient of $\mathbf{X}^{10}\mathbf{Y}^{10}$ in the expansion of $\mathbf{M}_2$.

P9 **(12 pts)** In the polynomial expansion of $\mathbf{M}_n$, find, with proof, the coefficient of $(\mathbf{XY})^k = \mathbf{XYXY}\dots\mathbf{XY}$ where k is a positive integer.

4 Rotations with Operator Exponentials (45 pts)

In this section, we will show that we can rotate vectors about axes using our noncommutative operators. First, let us explain the geometric problem: suppose that we are given the components of some vector $\vec{a}$ and we rotate that vector about the $\hat{n}$ axis an angle θ. Call the resulting vector $\vec{b}$. What are the components of $\vec{b}$?



[1] Technically, polynomials have only a finite number of terms. The appropriate description of what we are doing is a formal power series expansion, but we will not be concerned with this distincition.

Before we can proceed, we need two definitions:

Generator of Rotations Suppose that $\vec{\mathbf{J}} = (\mathbf{J}_1, \mathbf{J}_2, \mathbf{J}_3)$ and $\vec{\mathbf{X}} = (\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3)$ are two triples of operators. We say that $\vec{\mathbf{J}}$ generates vector rotations on $\vec{\mathbf{X}}$ if

1. $(\mathbf{J}_1, \mathbf{J}_2, \mathbf{J}_3)$ forms an angular momentum algebra.
2. $[\mathbf{J}_i, \mathbf{X}_j] = i\sum_{k=1}^{3} \epsilon_{i,j,k}\mathbf{X}_k$. Note the i in front of the sum is $\sqrt{-1}$ and the subscript $_i$ is $1, 2, 3$.
3. $\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3$ are linearly independent.

In order to use the commutators, let us first associate the vectors $\vec{a} = (a_1, a_2, a_3)$ and so on with operators:

$$\begin{aligned} \vec{a} &= (a_1, a_2, a_3) & \leftrightarrow && \vec{a} \cdot \vec{\mathbf{X}} &= a_1\mathbf{X}_1 + a_2\mathbf{X}_2 + a_3\mathbf{X}_3 \\ \hat{n} &= (n_1, n_2, n_3) & \leftrightarrow && \hat{n} \cdot \vec{\mathbf{J}} &= n_1\mathbf{J}_1 + n_2\mathbf{J}_2 + n_3\mathbf{J}_3 \\ \vec{b} &= (b_1, b_2, b_3) & \leftrightarrow && \vec{b} \cdot \vec{\mathbf{X}} &= b_1\mathbf{X}_1 + b_2\mathbf{X}_2 + b_3\mathbf{X}_3 \end{aligned}$$

where $n_1^2 + n_2^2 + n_3^2 = 1$ and $\vec{\mathbf{J}}$ generates vector rotations on $\vec{\mathbf{X}}$. We will prove that $\vec{a}$ can be rotated about $\hat{n}$ an angle of θ using the following equation:

$$\exp(-i\theta\hat{n} \cdot \vec{\mathbf{J}})\vec{a} \cdot \vec{\mathbf{X}} \exp(i\theta\hat{n} \cdot \vec{\mathbf{J}}) = \vec{b} \cdot \vec{\mathbf{X}} \tag{14}$$

Let us begin with some problems. Be sure to use the definition of the exponential of an operator.

P10 **(7 pts)** Prove that $\exp((s + t)\mathbf{A}) = \exp(s\mathbf{A})\exp(t\mathbf{A})$.

P11 The Hadamard Lemma:

(a) **(7 pts)** Suppose that $\mathbf{B}_0 = \mathbf{B}$ and that $\mathbf{B}_{n+1} = [\mathbf{A}, \mathbf{B}_n]$ for $n \geq 0$. Prove the following lemma for $N \geq 1$.

$$[\mathbf{A}^N, \mathbf{B}_0] = \sum_{j=1}^{N} \binom{N}{j} \mathbf{B}_j \mathbf{A}^{N-j} \tag{15}$$

(b) **(4 pts)** Prove that[2]

$$[\exp(\mathbf{A}), \mathbf{B}_0] = \sum_{j=1}^{\infty} \frac{\mathbf{B}_j}{j!} \exp(\mathbf{A}) \tag{16}$$

(c) **(3 pts)** Prove that

$$\exp(\mathbf{A})\mathbf{B}\exp(-\mathbf{A}) = \sum_{j=0}^{\infty} \frac{\mathbf{B}_j}{j!} \tag{17}$$

P12 **(7 pts)** Suppose that $\vec{\mathbf{J}}$ generates rotations on $\vec{\mathbf{X}}$. Use the Hadamard Lemma to prove that

$$\exp(-i\theta\mathbf{J}_3)[x_1\mathbf{X}_1 + x_2\mathbf{X}_2 + x_3\mathbf{X}_3] \exp(i\theta\mathbf{J}_3) = a_1\mathbf{X}_1 + a_2\mathbf{X}_2 + a_3\mathbf{X}_3 \tag{18}$$

where $(a_1, a_2, a_3) = (x_1 \cos\theta - x_2 \sin\theta, x_1 \sin\theta + x_2 \cos\theta, x_3)$. The following identities might be helpful:

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \ldots \qquad \sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \ldots \tag{19}$$

[2]Ignore issues of convergence.

P13 (a) **(7 pts)** Starting with Eq. (14), prove that

$$(\vec{b}-\vec{a})\cdot\vec{\mathbf{X}} = \theta(\hat{n}\times\vec{a})\cdot\vec{\mathbf{X}} + \sum_{k=2}^{\infty}\theta^k\frac{\vec{a}_k}{k!}\cdot\vec{\mathbf{X}} \tag{20}$$

where $\vec{a}_k = \hat{n}\times\vec{a}_{k-1}$ and $\vec{a}_0 = \vec{a}$ and the dot product $\vec{v}\cdot\vec{\mathbf{X}}$ means $v_1\mathbf{X}_1 + v_2\mathbf{X}_2 + v_3\mathbf{X}_3$.

(b) **(6 pts)** Prove that

$$\left|\sum_{k=2}^{\infty}\theta^k\frac{\vec{a}_k}{k!}\right| \le |\vec{a}|\theta^2 e^{|\theta|} \tag{21}$$

where $|\vec{v}| = \sqrt{v_1^2+v_2^2+v_3^2}$ denotes the magnitude of the vector $\vec{v}$.

P14 **(4 pts)** Give a geometric proof for why $\vec{b}-\vec{a}\approx\theta(\hat{n}\times\vec{a})$ when θ is much less than one. Clearly identify what approximation(s)[3] you used in order to demonstrate this result.

[3]While you may initially be uncomfortable with the idea of doing an approximation, being able to make appropriate approximations to get the correct intuition is an essential skill that you will develop if you continue to study math and physics at the college level. Unfortunately, when many students first study math, rigor is emphasized over intution and students acquire an inflexibility in their ability to think differently.