1. Given that $2 \times 10 \times 101 = 2020$, compute $4 \times 5 \times 303$.

   Answer: 6060

   Solution: Note that $4 \times 5 = 2 \times 10$ and $101 \times 3 = 303$. Therefore, $4 \times 5 \times 303 = 3 \times (2 \times 10 \times 101) = 3 \times 2020 = \boxed{6060}$.

2. Ariel the Frog is on the top left square of a 8×10 grid of squares. Ariel can jump from any square on the grid to any adjacent square, including diagonally adjacent squares. What is the minimum number of jumps required for Ariel to reach the bottom right corner?

   Answer: 9

   Solution: If Ariel wants to reach the bottom corner with the minimum number of jumps, she should jump diagonally each time she has the opportunity to. She can do so until the very last row, where the bottom right square is 2 jumps away. It takes her 7 jumps to jump from the top left square to the bottom row (diagonally) and 2 more jumps to get to the destination. Therefore, it takes her $7 + 2 = \boxed{9}$ total jumps.

3. When Star tries to split his gold bars into groups of 4, he has 3 left over, and when he tries to split his gold bars into group of 5, he has 4 left over. What is the least number of gold bars Star could have?

   Answer: 19

   Solution: Let S be the number of gold bars Star has. S must be 1 less than a multiple of 4 and a multiple of 5. Thus, $S + 1$ is a multiple of both 4 and 5, and $20 \mid (S + 1)$. The smallest S that satisfies this is $S = \boxed{19}$.

4. As an afternoon activity, Emilia will either play exactly two of four games (Minecraft, Fortnite, Undertale, and Fire Emblem) or work on homework for exactly one of three classes (Physics 5A, Math 1B, and Anthropology 3). How many choices of afternoon activities does Emilia have?

   Answer: 9

   Solution: There are $\binom{4}{2} = 6$ choices of games, and 3 choices of homework, so in total there are $6 + 3 = \boxed{9}$ choices of afternoon activities.

5. Matthew wants to buy merchandise of his favorite sports team. He wants to buy posters of the players on the team, but he only has 30 dollars to spend. If he can buy 2 posters for 4 dollars and 5 posters for 8 dollars, what is the maximum number of posters that Matthew can buy?

   Answer: 17

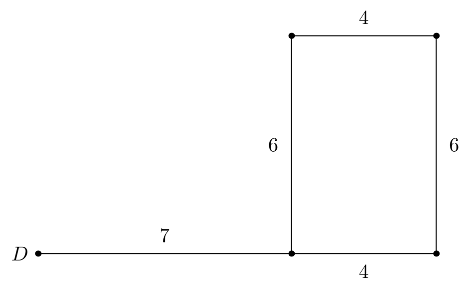
   Solution: Note Matthew can buy at most 5 posters with every 8 dollars, so the maximum number of posters that he can get is $5 \times 3 + 2 = \boxed{17}$.

6. Ada draws six lines in a plane such that no two lines are parallel and no three lines all intersect at the same point. What is the maximum number of regions she can divide the plane into? One line divides the plane into two regions.

   Answer: 22

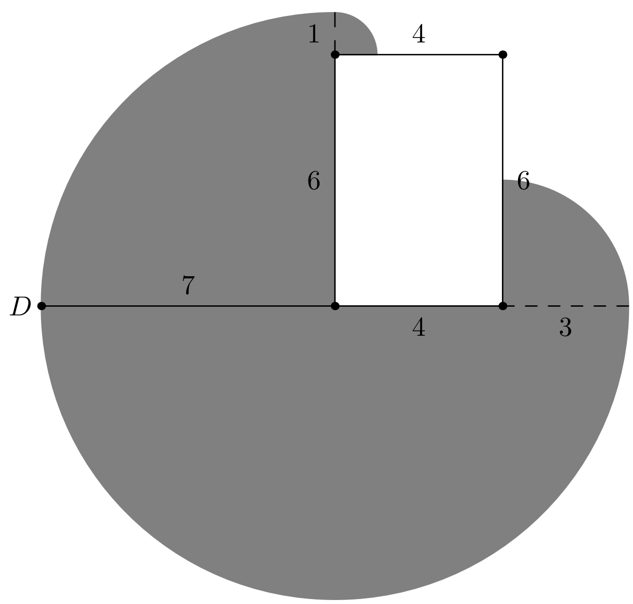
   Solution: Note that a line intersecting with $k - 1$ lines creates k new regions. Therefore, n lines can divide a plane into at most $1 + (1 + 2 + \cdots + n) = \frac{n(n+1)}{2} + 1$ regions, and our answer is $\frac{6 \cdot 7}{2} + 1 = \boxed{22}$.

7. Deepak the Dog is tied with a leash of 7 meters to a corner of his 4 meter by 6 meter rectangular shed such that Deepak is outside the shed. Deepak cannot go inside the shed, and the leash cannot go through the shed. Compute the area of the region that Deepak can travel to.



Answer: $\mathbf{\frac{157\pi}{4}}$ **OR** $\mathbf{39.25\pi}$ **OR** $\mathbf{39\frac{1}{4}\pi}$

Solution:

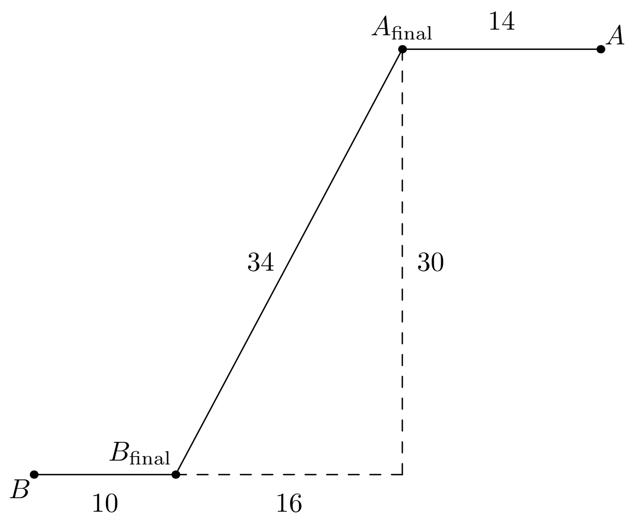


The desired region is indicated as the shaded part in the diagram, which consists of a quarter circle of radius 1, a quarter circle of radius 3, and a three-quarters circle of radius 7. The total area is then $\frac{1}{4}\pi \cdot 1^2 + \frac{1}{4}\pi \cdot 3^2 + \frac{3}{4}\pi \cdot 7^2 = \boxed{\frac{157\pi}{4}}$.

8. City A is 30 miles north and 40 miles east of city B. Alice leaves from city A and drives west at a constant speed of 35 miles per hour at the same time as Bob leaves from city B and drives east at 25 miles per hour. How many minutes will it take for them to be 34 miles apart for the first time?

Answer: 24

Solution:



Alice will always be 30 miles north of Bob, so by Pythagorean Theorem, to be 34 miles apart from Bob she must also be $\sqrt{34^2 - 30^2} = 16$ miles either east or west from Bob. Since they start 40 miles apart, they must travel a total of $40 - 16 = 24$ miles to be 16 miles apart. They are traveling towards each other horizontally at a speed of $35 + 25 = 60$ miles per hour, so it will take them $\boxed{24}$ minutes to be 16 miles apart horizontally and 34 miles apart in total.

9. The quadratic equation $a^2x^2 + 2ax - 3 = 0$ has two solutions for x that differ by a, where $a > 0$. What is the value of a?

   Answer: 2

   Solution: The above equation can be factored as $(ax + 3)(ax - 1) = 0$, so $x = -\frac{3}{a}$ or $x = \frac{1}{a}$ as $a \neq 0$. The difference in the solutions is then $\frac{4}{a} = a$. Given $a > 0$, $a = \boxed{2}$.

10. Find the number of ways to color a 2×2 grid of squares with 4 colors such that no two (non-diagonally) adjacent squares have the same color. Each square should be colored entirely with one color. Colorings that are rotations or reflections of each other should be considered different.

    Answer: 84

    Solution: Consider the two squares in the top-right and the bottom-left of the grid. If these two squares have the same color, then we have 4 possibilities for the color of these two squares and $3 \cdot 3$ possibilities for the color of the other two squares. If the top-right and the bottom-left squares have different colors, then we have $4 \cdot 3$ ways to pick these two squares and $2 \cdot 2$ ways to pick the other two squares. Thus, the total number of ways is $4 \cdot 3 \cdot 3 + 4 \cdot 3 \cdot 2 \cdot 2 = \boxed{84}$.

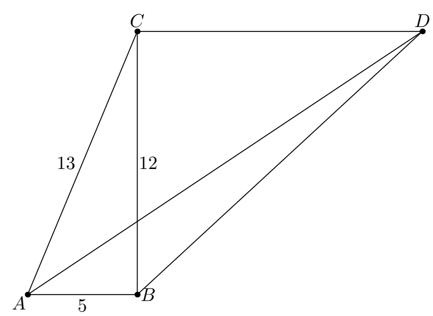
11. Let x and y be real numbers such that $xy = 4$ and $x^2y + xy^2 = 25$. Find the value of $x^3y + x^2y^2 + xy^3$.

    Answer: $\mathbf{\frac{561}{4}}$ **OR** $\mathbf{140\frac{1}{4}}$ **OR 140.25**

Solution: Since $xy = 4$ and $xy(x+y) = x^2y + xy^2 = 25$, we have $x + y = \frac{25}{4}$. Thus,

$$\begin{aligned} x^3y + x^2y^2 + xy^3 &= xy(x^2 + xy + y^2) \\ &= xy((x+y)^2 - xy) \\ &= 4 \cdot \left(\left(\frac{25}{4} \right)^2 - 4 \right) \\ &= \boxed{\frac{561}{4}}. \end{aligned}$$

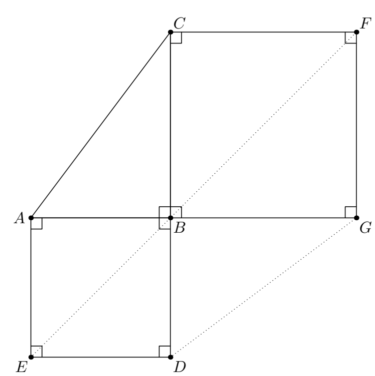
12. Right triangle ABC has $AB = 5$, $BC = 12$, and $CA = 13$. Point D lies on the angle bisector of $\angle BAC$ such that CD is parallel to AB. Compute the length of BD.



Answer: $\mathbf{\sqrt{313}}$

Solution: Notice that because CD is parallel to AB, $\angle CDA = \angle BAD = \angle DAC$. Thus, $AC = CD = 13$, so $BC = \sqrt{BC^2 + CD^2} = \sqrt{12^2 + 13^2} = \boxed{\sqrt{313}.}$

13. Let $\triangle ABC$ be a right triangle with $m\angle B = 90°$ such that AB and BC have integer side lengths. Squares $ABDE$ and $BCFG$ lie outside $\triangle ABC$. If the area of $\triangle ABC$ is 12, and the area of quadrilateral $DEFG$ is 38, compute the perimeter of $\triangle ABC$.



Answer: $\mathbf{10 + 2\sqrt{13}}$

Solution: Let $AB = a$, $BC = b$. Then $ab = 2 \cdot A_{\triangle ABC} = 24$. Note that $38 = A_{DEFG} = \frac{a^2}{2} + \frac{b^2}{2} + \frac{ab}{2}$, as the interior of $DEFG$ is the union of half of the two squares and a right triangle with leg lengths a and b. Thus $a^2 + b^2 = 52$, and $\sqrt{a^2 + b^2} = \sqrt{52} = 2\sqrt{13}$ is the hypotenuse length of $\triangle ABC$. It follows that $(a,b) = (4,6)$ or $(a,b) = (6,4)$, and $P_{\triangle ABC} = \boxed{10 + 2\sqrt{13}}$.

14. Shivani is planning a road trip in a car with special new tires made of solid rubber. Her tires are cylinders that have width 6 inches and have diameter 26 inches, but need to be replaced when the diameter is less than 22 inches. The tire manufacturer claims that 0.12π cubic inches of its tire will wear away with every single rotation. Assuming that the tire manufacturer is correct about the wear rate of its tires, and that the tire loses volume by reducing radius **ONLY**, how many revolutions can each tire make before Shivani needs to replace it?

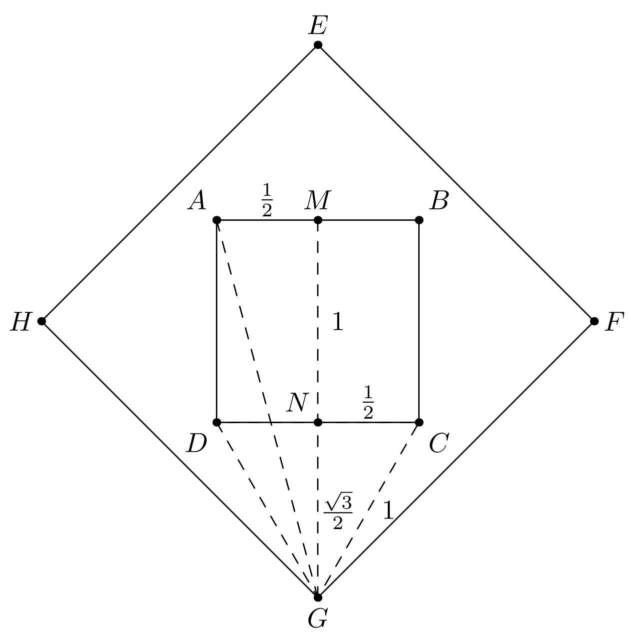
    Answer: 2400

    Solution: Notice that the volume of the tire that has worn away equals the number of times it rotated, multiplied by the rate of wear per rotation. Each tire is replaced before wearing off $((\frac{26}{2})^2\pi - (\frac{22}{2})^2\pi) \cdot 6 = 288\pi$ cubic inches, by which time it has made R revolutions. Then $288\pi = 0.12\pi \cdot R$, so $R = \boxed{2400}$.

15. Square $ABCD$ has side length 1 and square $EFGH$ has side length greater than 1. $ABCD$ and $EFGH$ share the same center, and ABE, BFC, CGD, and DHA are all equilateral triangles. If x is the length of AG, find x^2.

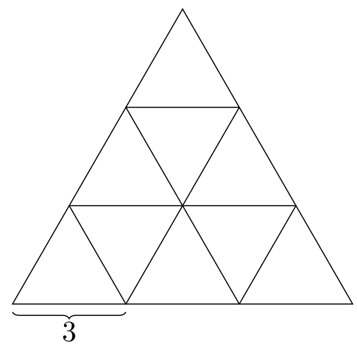
    Answer: $\mathbf{2 + \sqrt{3}}$

    Solution:



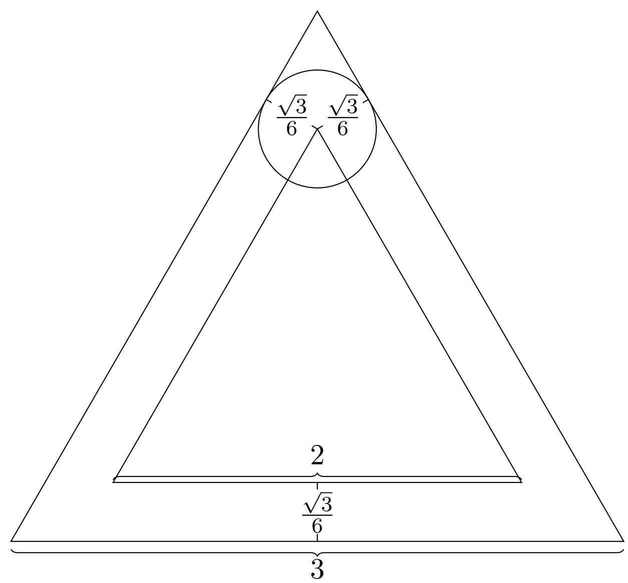
    Let M be the midpoint of $\overline{AB}$ and N be the midpoint of $\overline{DC}$. Then $NC = \frac{1}{2}$ by definition of a midpoint. Also since triangle DCG is equilateral with side length 1, $CG = 1$. Thus by Pythagorean theorem on triangle CNG, $NG = \sqrt{CG^2 - NC^2} = \frac{\sqrt{3}}{2}$. Since $MN = 1$, we have that $MG = 1 + \frac{\sqrt{3}}{2}$. Finally, since $AM = \frac{1}{2}$, we have by Pythagorean theorem on triangle AMG that $x^2 = AG^2 = AM^2 + MG^2 = \frac{1}{4} + (\frac{7}{4} + \sqrt{3}) = 2 + \sqrt{3}$ as desired.

16. Rohith has a large equilateral triangle grid consisting of 9 equilateral triangles, where each small triangle has side length 3. He also has a circular coin with a diameter of $\frac{\sqrt{3}}{3}$. Given that he randomly throws the coin such that the center of the coin lies in the large triangular grid, find the probability that the coin lands completely inside a small equilateral triangle.



Answer: $\frac{4}{9}$

Solution:



For the coin to lie completely inside a equilateral triangle of side length 2, the center of the coin must be at least distance $\frac{\sqrt{3}}{6}$ away from all three sides. Note that the region that the coin can lie in is an equilateral triangle of side length 2 contained within the equilateral triangle of side length 3. Thus our probability is $\left(\frac{2}{3}\right)^2 = \boxed{\frac{4}{9}}$.

17. Let $\{a_i\}$ for $1 \leq i \leq 10$ be a finite sequence of 10 integers such that for all odd i, $a_i = 1$ or -1, and for all even i, $a_i = 1, -1$, or 0. How many sequences $\{a_i\}$ exist such that $a_1 + a_2 + a_3 + \cdots + a_{10} = 0$?

Answer: 1052

Solution: If k elements of the sequence equal 0, then k must be even in order for the sum to equal 0, so $k = 0, 2, 4$. In that case, there must be $\frac{10-k}{2}$ each of 1's and -1's among the sequence. There are $\binom{5}{k}$ ways to pick which k elements of the sequence are 0 and $\binom{10-k}{\frac{10-k}{2}}$ ways to pick which elements of the sequence are 1's. Then the rest are -1's. Thus, the answer is the sum over $k = 0, 2, 4$ over the binomial coefficient

$$\binom{5}{k}\binom{10-k}{\frac{10-k}{2}}.$$

We have

$$\binom{5}{0}\binom{10}{5}+\binom{5}{2}\binom{8}{4}+\binom{5}{4}\binom{6}{3}=252+700+100=\boxed{1052}.$$

18. What is the smallest positive integer x such that there exists an integer y with $\sqrt{x}+\sqrt{y}=\sqrt{1025}$?

Answer: 41

Solution: Note that $\sqrt{1025}=5\sqrt{41}$. Subtracting $\sqrt{x}$ from both sides and squaring both sides gives $y=1025+x-2\cdot 5\sqrt{41x}$. Therefore, $\sqrt{41x}$ must be an integer, and $41 \mid x$. Then the smallest positive value of x is $\boxed{41}$, in which case $y=16\cdot 41$.

19. Let

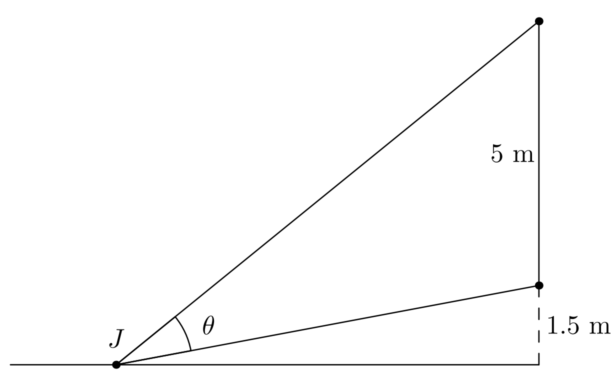
$$a=\underbrace{19191919\ldots1919}_{\text{19 is repeated 3838 times}}.$$

What is the remainder when a is divided by 13?

Answer: 6

Solution: Note that $13 \mid 191919$. Thus, $13 \mid \underbrace{19191919\ldots191900}_{\text{19 is repeated 3837 times}}$, since $3 \mid 3837$. However, $a=\underbrace{19191919\ldots191900}_{\text{19 is repeated 3837 times}}+19$, so the remainder when a is divided by 13 is the same as the remainder when 19 is divided by 13, or $\boxed{6}$.

20. James is watching a movie at the cinema. The screen is on a wall perpendicular to the floor and is 5 meters tall with the bottom edge of the screen 1.5 meters above the floor. James wants to find a seat which maximizes his vertical viewing angle (depicted below as θ in a two dimensional cross section), which is the angle subtended by the top and bottom edges of the screen. How far back from the screen in meters (measured along the floor) should he sit in order to maximize his vertical viewing angle?



Answer: $\frac{\sqrt{39}}{2}$ **OR** $\sqrt{\frac{39}{4}}$

Solution: Imagine a 2D cross-section of the cinema containing James and perpendicular to the screen. Draw a circle through James, the top edge of the screen and the bottom edge of the screen. The vertical viewing angle is maximized when the circle is tangent to the floor. This is because the radius of the circle is minimized when that happens and given that the viewing angle is twice the angle of the arc that starts from the top of the screen to the bottom of the screen, since the distance from the top edge of the screen to the bottom edge of the screen is fixed, the angle measurement of the arc is maximized when the radius of the circle is minimized. Then by the power of the point theorem the distance is $\sqrt{1.5 \times 6.5} = \boxed{\dfrac{\sqrt{39}}{2}}$.

What follows is a proof that the radius of the circle is minimized when the circle is tangent to the floor. Let ℓ be the perpendicular bisector of the top of the screen and the bottom of the screen, let O be a point on this line, let C be the circle with center O that passes through the top and bottom edges of the screen and let A be the arc starting from the point closest to and passes through the bottom of the screen and the top of the screen, in that order. Note that the vertical viewing angle is half the measure of the arc A. Now imagine moving the point O toward James starting from behind the screen. As we do this, the arc A gets larger, the radius of C gets larger and the angle subtended by the screen from a point on A gets smaller. Then the maximum angle occurs when the arc A first touches the floor, which is when the circle C is tangent to the floor.