1. If an integer is divisible by 9, then it is divisible by 3.

   Answer: True

   Solution: Since $9 = 3 \cdot 3$, a number divisible by 9 must also be divisible by 3; the answer is $\boxed{\text{true}}$.

2. There exists a real number x that does not satisfy the following inequality: $|8 - 4x| \geq -2$.

   Answer: False

   Solution: According to the definition of absolute value, $|8 - 4x| \geq 0$. Therefore, $|8 - 4x| \geq -2$ for all real numbers x, so the answer is $\boxed{\text{false}}$.

3. If $ab = ac$ and a, b, c are integers, then b must equal c.

   Answer: False

   Solution: Let $a = 0$, $b = 1$, and $c = 2$. Then $ab = ac$ but $b \neq c$, so the answer is $\boxed{\text{false}}$.

4. If a, b, c are positive integers, then it is always true that $||a - b| - c| = |a - |b - c||$.

   Answer: False

   Solution: When $a = 2, b = 1, c = 1$, we have $||a-b|-c| = |1-1| = 0$ but $|a-|b-c|| = |2-0| = 2$. Thus the answer is $\boxed{\text{false}}$.

5. If a right triangle has a side of length a and a side of length b, then its area is always $\frac{1}{2}ab$.

   Answer: False

   Solution: Consider a 3-4-5 right triangle with $a = 3$ and $b = 5$. The area of the triangle is $\frac{1}{2} \cdot 3 \cdot 4 = 6$ but $\frac{1}{2} \cdot 3 \cdot 5 = \frac{15}{2}$. Thus the answer is $\boxed{\text{false}}$.

6. There exist integers (a, b, c, n) such that $a^n + b^n = c^n$.

   Answer: True

   Solution: Take $a = b = c = 0$ and $n = 1$. Then $0^1 + 0^1 = 0^1$, so the answer is $\boxed{\text{true}}$.

7. The probability of obtaining an even number of heads is less than the probability of obtaining an odd number of heads when flipping 5 coins.

   Answer: False

   Solution: Since the probability of flipping a heads is the same as flipping a tails, the probability of obtaining 0 heads is the same as flipping 0 tails, or 5 heads. By the same logic, the probabilities of flipping 1 heads and 4 heads are the same, and the probability of flipping 2 heads and 3 heads are the same. Thus the probabilities of obtaining an even number of heads and obtaining an odd number of heads are equal, and the answer is $\boxed{\text{false}}$.

8. 2020 has exactly 4 positive factors that are 1 more than a perfect square.

   Answer: False

   Solution: The positive factors of 2020 are 1, 2, 4, 5, 10, 20, 101, 202, 404, 505, 1010, and 2020. Of these factors, $1 = 0^2 + 1$, $2 = 1^2 + 1$, $5 = 2^2 + 1$, $10 = 3^2 + 1$, and $101 = 10^2 + 1$ are one more than a perfect square, and there are 5 such factors, so the answer is $\boxed{\text{false}}$.

9. There exist 10 distinct non-negative integers that sum to 43.

   Answer: False

   Solution: The sum of the 10 smallest distinct non-negative integers is $0+1+2+3+4+5+6+7+8+9=45$, so any sum of 10 distinct non-negative integers is at least 45. Since $43<45$, no such integers exist, and the answer is $\boxed{\text{false}}$.

10. 2^{100} is more than 30 digits long when written in base 10.

    Answer: True

    Solution: Note that $2^{10}>1000=10^3$. Then $2^{100}>10^{3\times10}=10^{30}$, and 10^{30} is at least 31 digits long. Thus 2^{100} is at least 30 digits long, and the answer is $\boxed{\text{true}}$.

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11. How many positive prime numbers are less than 3?

    Answer: 1

    Solution: The only positive prime number less than 3 is 2, so the answer is $\boxed{1}$.

12. What is the perimeter of a regular octagon with side length 5?

    Answer: 40

    Solution: The perimeter is $5\times8=\boxed{40}$.

13. Compute $(5-2)^2\times3-6$.

    Answer: 21

    Solution: Using order of operations, we have

    $$(5-2)^2\times3-6=3^2\times3-6=9\times3-6=27-6=\boxed{21}.$$

14. If $x=y+z$ and $x+y=z$, compute the value of y.

    Answer: 0

    Solution: Rearranging the equations gives $x-z=y$ and $x-z=-y$. It must follow that $y=-y$ and thus $y=\boxed{0}$.

15. How many positive factors does 205 have?

    Answer: 4

    Solution: The prime factorization of 205 is 5×41. Thus, the positive factors of 205 are $1,5,41$ and 205. This gives a total of $\boxed{4}$ positive factors.

16. A school with 8 classes is going on a field trip. Each class has 20 students and 1 teacher. What is the fewest number of 40-passenger buses needed to transport the 8 classes?

    Answer: 5

    Solution: There are a total of $21\times8=168$ passengers. Since $4<168/40<5$, we need $\boxed{5}$ buses to seat them all.

17. Compute the sum of the first 10 positive integer multiples of 3.

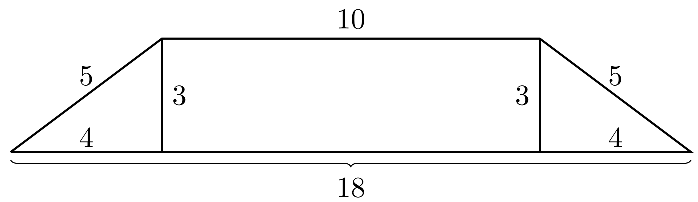
    Answer: 165

    Solution: This sum equals $3 \cdot (1 + 2 + \cdots + 10) = 3 \cdot \frac{10 \cdot 11}{2} = 3 \cdot 55 = \boxed{165}$.

18. What is the area of a trapezoid with side of length 5, 10, 5, and 18 if the sides of length 10 and 18 are parallel?

    Answer: 42

    Solution:



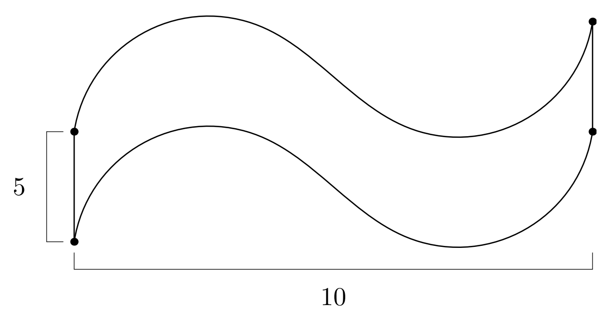
    The difference in the length of the bases is 8. Because the trapezoid is isosceles, we can draw 2 altitudes to divide the trapezoid into two $3-4-5$ triangles and one 3 by 10 rectangie, so the height of the trapezoid is 3. Thus the area is $3 \cdot (10 + 18)/2 = \boxed{42}$.

19. The measure of the largest angle in triangle ABC is 132 degrees. Let EFG be a triangle such that $EF = 2AB$, $FG = 2BC$, and $GE = 2CA$. What is the degree measure of the largest angle of triangle EFG?

    Answer: 132

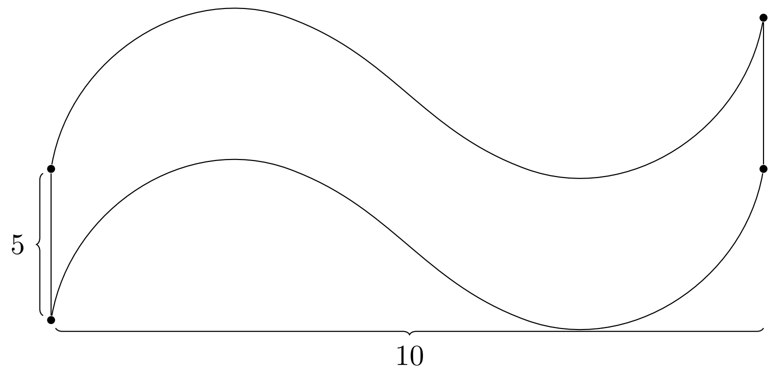
    Solution: Note that the triangle's angles don't change when all the sides are scaled by the same amount. The answer is thus $\boxed{132}$.

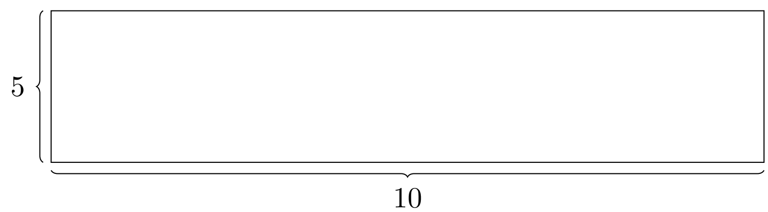
20. What is the area of the S-shaped figure below, which has constant vertical height 5 and width 10?



    Answer: 50

    Solution:





This is Archimedes' principle! Each of the vertical strips lying in the S-shaped figure is of length 5. When we flatten the figure to a rectangle of height 5 and width 10, the area will remain the same. Thus, the area is $5 \cdot 10 = \boxed{50}$.

21. Find the integer closest to $4 \cdot \sqrt{14}$.

    Answer: 15

    Solution: We note that the square of $4 \cdot \sqrt{14}$ is $16 \cdot 14 = 224$. The closest perfect square to 224 is $225 = 15^2$, thus $\boxed{15}$ is the closest integer.

22. A computer is priced at 1000 dollars. Jillian has a 25 percent off discount and a 50 dollar off discount, and can use both discounts in either order. What is the lowest price in dollars that Jillian can pay for the computer?

    Answer: 700

    Solution: Jillian pays $0.75 \cdot 1000 - 50 = 700$ dollars when using the percent discount first and dollar discount second, and pays $0.75(1000 - 50) = 712.5$ dollars when using the discounts in the other order. Thus the lower amount Jillian can pay is $\boxed{700}$.

23. Geometric sequences $\{a_n\}$ and $\{b_n\}$ have the same common ratio. If $a_1 = 2b_1$, compute $\frac{a_{20}}{b_{20}}$.

    Answer: 2

    Solution: This is just $\boxed{2}$, since the ratio of the corresponding terms remains constant (the common ratios cancel).

24. Two consecutive perfect squares differ by 15. What is the positive square root of the larger of the two perfect squares?

    Answer: 8

    Solution: Let x be the positive square root of the larger perfect square. Thus, we have that $(x)^2 - (x-1)^2 = 2x - 1 = 15$, so $x = \boxed{8}$.

25. Joey and Austin each individually pick an integer from 0 to 9, inclusive. Joey wins if the product of the two numbers is 0, and Austin wins otherwise. Assuming both players play optimally, what is the probability Joey wins?

    Answer: 1 or 100%

    Solution: Joey will pick 0 every time to make the product of the two numbers 0, so he wins with probability $\boxed{1}$.

26. What is the minimum number of coins needed to make 87 cents using pennies, nickels, dimes, and quarters? A penny is 1 cent, a nickel is 5 cents, a dime is 10 cents, and a quarter is 25 cents.

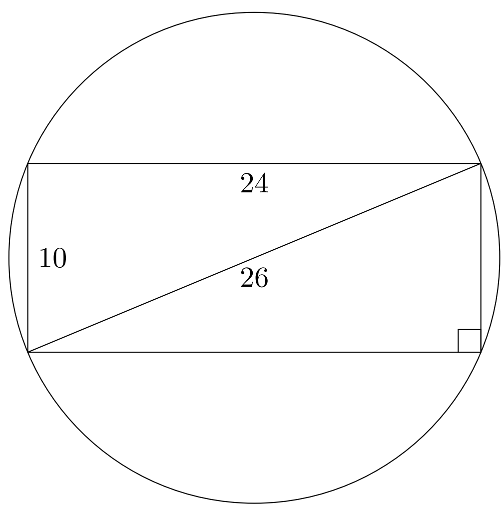
    Answer: 6

    Solution: The minimum number of coins is obtained when we use three quarters, one dime, and two pennies. This is optimal: we need 2 pennies since all the other coins are multiples of 5 cents, using 1 dime is strictly better than using 2 nickels, and using 1 quarter is strictly better than using 2 dimes and a nickel. Thus we need a minimum of $\boxed{6}$ coins.

27. A rectangle of height 10 and width 24 is inscribed in a circle. What is the circumference of that circle? Express your answer in terms of π.

    Answer: 26π

    Solution:



The diagonal of the rectangle is 26, since it is the hypotenuse of a right triangle with legs 10 and 24. The diagonal is also the diameter of the circle since any right triangle inscribed in a circle has a diameter as its hypotenuse, so the circumference is $\boxed{26\pi}$.

28. Three teams each with three people decide to shake hands. If everyone does not shake his or her teammates' hands, but shakes hands with everyone else exactly once, how many handshakes occur?

    Answer: 27

    Solution: There are three pairs of teams. Each pair of teams will have $3 \times 3 = 9$ handshakes. The answer is thus $\boxed{27}$, as each pair of teams will have 9 handshakes between them.

29. Dylan takes marbles from a bag with 2 red marbles, 2 green marbles, 3 blue marbles, and 4 purple marbles. If Dylan takes marbles out of the bag without replacement, and does not look at the marbles he takes, how many marbles must Dylan take to ensure that he has at least three different colors of marbles?

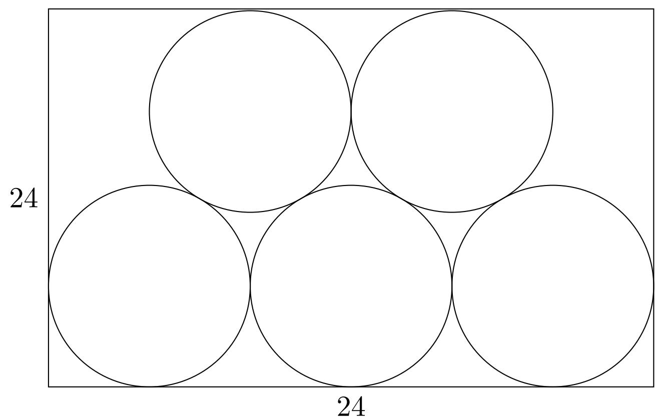
    Answer: 8

    Solution: In the worst case, Dylan will take all of the purple and blue marbles before he draws a third color. Dylan thus needs to draw $3+4+1 = \boxed{8}$ marbles to ensure at least three different colors of marbles are taken.

30. What's the maximum number of circles of radius 4 that fit into a 24×15 rectangle without overlap?

    Answer: 5

    Solution:



A quick check shows that it is possible to fit 5 circles inside this rectangle; put three circles at the bottom of the rectangle, and there is space for two more circles in the space left over. Now, we will show that 6 circles is impossible. Note that the vertical distance between the centers of any two circles is at most $15-4-4 = 7$. The distance between any two centers is at least $4+4 = 8$, so the horizontal distance between the centers of any two circles is at least $\sqrt{8^2-7^2} = \sqrt{15}$. The leftmost and rightmost centers can be at most $24-4-4 = 16$ apart; as such, the total horizontal distance that the circles take if there are 6 circles is at least $5 \cdot \sqrt{15}+8 > 24$. Thus, the answer is $\boxed{5}$.

31. What is the surface area of a rectangular prism with volume 30 if it has a square base and all edge lengths are integers?

    Answer: 122

    Solution: Since 1 is the only perfect square factor of 30, the base must be 1×1. Thus, the height of the prism is 30, and the surface area is $30 \cdot 1 \cdot 4 + 1 \cdot 1 \cdot 2 = \boxed{122}$.

32. John received the scores of 80, 87, 91, 93, and 100 on his past 5 tests. If John can only receive integer scores between 0 and 100 inclusive, how many ordered pairs of scores (a, b) are there for his last two tests such that the average of his seven test scores is exactly 93?

    Answer: 1

Solution: We notice the five scores are 13 below, 6 below, 2 below, equal to, and 7 above the desired average of 93. Combined, they are a total of 14 below the average. For the average of all seven scores to be 93, the remaining two scores must be a total of 14 above the average. Since the maximum score is 100, the only way to do this is if both scores are 7 above the average; i.e. both scores are 100. Thus there is only $\boxed{1}$ ordered pair, $(7, 7)$.

33. Tim is currently three times as old as Tom. In 10 years, Tim will be twice as old as Tom. How old is Tom currently?

Answer: 10

Solution: Let x be Tom's age, so Tim's age is $3x$. In 10 years, their ages will be $x + 10$ and $3x + 10$, respectively. Thus, $3x + 10 = 2(x + 10)$, and we find that Tom's age is $x = \boxed{10}$.

34. Thirty people are equally spaced around a circular table and are numbered 1 through 30 going counterclockwise. What is the degree measure of the angle formed by the people numbered 1, 15, and 4 where the vertex of the angle is at the person numbered 15?

Answer: 18

Solution: Notice that the measure of the arc between any two consecutive people at the table is $\frac{360^\circ}{30} = 12^\circ$. Thus, the length of the arc enclosed by persons 1 and 4 is $12^\circ \cdot 3 = 36^\circ$, and the inscribed angle is half of that, or $\boxed{18}$ degrees.

35. What is the smallest positive integer that is divisible by 44 and has only ones and zeros in its base-ten representation?

Answer: 1100

Solution: For a number to be divisible by 44, it must be divisible by both 11 and 4. For the number to be divisible by 4, it must end in at least two zeros. Furthermore, the number of zeros at the end of a number does not affect whether the number is divisible by 11. The smallest positive number divisible by 11 is 11, thus our answer is 11 with two zeros appended to it, or $\boxed{1100}$.

36. At BMT 2018, Eric is taking two focus (individual) rounds, each of which consists of 5 questions. In how many different ways can he get exactly 5 of the 10 total problems correct among both tests, up to the specific problems he gets right and wrong on each test? (For example, if Eric gets the first four correct on the first test and only the first question on the second test, this is different from getting the last four correct on the first test and only the first question on the second test).

Answer: $\boldsymbol{\binom{10}{5}}$ **or 252**

Solution: Since all 10 problems are distinct, the problem statement is equivalent to counting the number of ways to get 5 problems right out of 10, and there are $\binom{10}{5} = \boxed{252}$ ways of doing so.

37. Phil's eighteen-wheeler truck has eighteen tires and six spare tires. Each tire has a circumference of 2 meters. Phil needs to make a 4800-meter trip and wants to make sure all 24 of his tires spin the same number of times. How many times will each tire spin?

Answer: 1800

Solution: Eighteen tires spin at a time, and $4800/2 = 2400$ spins are needed to go 4800 meters, so the total number of spins is $18 \cdot 2400$. Thus each tire spins $18 \cdot 2400/24 = \boxed{1800}$ times.

38. How many positive two-digit integers are 45 more than their reverse? For example, 90 is 81 more than its reverse 09.

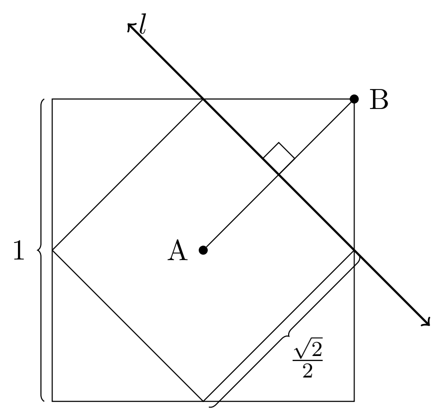
Answer: 5

Solution: The two digit number $\underline{A}\,\underline{B}$ is equal to $10A+B$, and its reverse is equal to $10B+A$. Their difference is then $9(A-B)=45$. Thus $A-B=5$, and there are $\boxed{5}$ integers satisfying this: 50, 61, 72, 83, 94.

39. In a square of area 1, what is the area of the region formed by the set of all points inside the square that are closer to the center of the square than to any of its four vertices?

Answer: $\frac{1}{2}$ OR 0.5

Solution:



First we need to know that for any two distinct points A, B in the plane, the perpendicular bisector ℓ of AB divides the plane into two regions. All points in the region containing A are closer to A than to B, and all points in the region containing B are closer to B than to A. Applying this reasoning here, the desired region is bound by the perpendicular bisectors of the 4 line segments from each vertex of the square to the square's center. Note that the desired region is a square of side length $\frac{\sqrt{2}}{2}$, so it must have area $\boxed{\frac{1}{2}}$.

40. Compute the sum of all irreducible fractions of the form $\frac{a}{34}$, where a is an integer between 1 and 33, inclusive.

Answer: 8

Solution: Notice that if $\frac{a}{34}$ is irreducible, then $\frac{34-a}{34}=1-\frac{a}{34}$ is also irreducible. Pairing up terms in this manner and observing that $\frac{a}{34}+\frac{34-a}{34}=1$, we see that the sum of all irreducible fractions is $\frac{n}{2}$, where n is the number of a's such that $\frac{a}{34}$ is irreducible (note that $\frac{17}{34}$ can be reduced to $\frac{1}{2}$ so we do not need to worry about the case where $a=34-a$). There are a total 16 possible a with $1 \le a \le 33$ such that $\frac{a}{34}$ is irreducible (all odd a except 17), so the answer is $\frac{16}{2}=\boxed{8}$.

41. An ant is at vertex A of cube $ABCD-EFGH$, where $ABCD$ is one of the square faces with AC as a diagonal. Each second, the ant randomly chooses a vertex adjacent to its current vertex to crawl towards. What is the probability the ant is on vertex C after 9 seconds?

Answer: 0

Solution: The ant can only be at vertex C after an even number of steps from vertex A, so it is impossible for the ant to be on vertex C after 9 steps, and our probability is $\boxed{0}$.

42. What is the sum of all integers k such that $\frac{2020^3}{k}$ is an integer?

Answer: 0

Solution: If $\frac{2020^3}{k}$ is an integer, so is $\frac{2020^3}{-k}$. Thus the sum over all such integers k is $\boxed{0}$.

43. Out of the eight interior angles of a convex octagon (an octagon where none of the interior angles are greater than 180°), at most how many of them can be 120°?

Answer: 5

Solution: The exterior angles of a octagon sum to 360°. The exterior angle of a 120° angle is 60°. Thus, there can be at most $\boxed{5}$ such angles before the sum exceeds 360°.

44. $P(x)$ and $Q(x)$ are two quadratic functions such that taking the difference between them results in a quadratic function with no zeroes. When graphed in the xy-plane, at how many points do P and Q intersect?

Answer: 0

Solution: For P and Q to intersect, their difference must be 0 at some point. But $P - Q$ has no real roots, so their difference is never zero, and they intersect at $\boxed{0}$ points.

45. Alice and Bob each roll a six-sided die, obtaining two (not necessarily distinct) numbers A and B. What is the probability that both $\underline{A}\,\underline{B}$ and $\underline{B}\,\underline{A}$ are prime (where $\underline{X}\,\underline{Y}$ is the two digit number with ones digit Y and tens digit X for $1 \le X \le 9$, $0 \le Y \le 9$)?

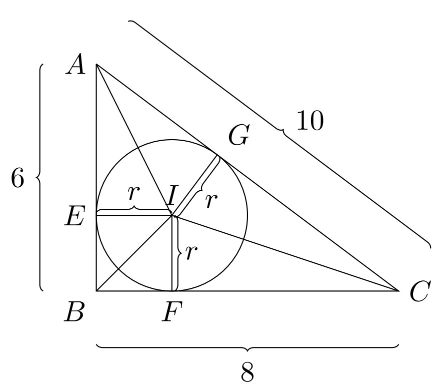
Answer: $\frac{1}{12}$ OR $0.08\underline{3}$

Solution: For a two-digit number to be prime, its ones digit must be $1, 3, 7$, or 9, but only 1 and 3 are possible outcomes from a six-sided die. Then the possible values for $\underline{A}\,\underline{B}$ are $11, 13, 31$, all of which are prime– 3 prime results out of $6 \cdot 6 = 36$ possible outcomes. Thus our answer is $\dfrac{3}{36} = \boxed{\dfrac{1}{12}}$.

46. What is the radius of a circle inscribed in a triangle with side lengths 6, 8, and 10?

Answer: 2

Solution:



The radius of a circle inscribed in a triangle is equal to twice the triangle's area divided by the triangle's perimeter. Here, this equals $(2 \cdot 24)/24 = \boxed{2}$.

For a proof of the above result that the radius of the inscribed circle in a triangle is equal to twice the triangle's area divided by the triangle's perimeter, let our triangle be ABC, our inscribed circle with center I and radius r be tangent to side AB at E, side BC at F, and side AC at G. Letting $[XYZ]$ denote the area of triangle XYZ, we have

$$\begin{aligned}
[ABC] &= [AGI] + [GIC] + [CIF] + [BFI] + [EIB] + [AIE] \\
&= \frac{1}{2}AG \cdot GI + \frac{1}{2}GC \cdot GI + \frac{1}{2}FC \cdot FI + \frac{1}{2}BF \cdot FI + \frac{1}{2}EB \cdot EI + \frac{1}{2}AE \cdot EI \\
&= \frac{1}{2}(AG \cdot r + GC \cdot r + FC \cdot r + BF \cdot r + EB \cdot r + AE \cdot r) \\
&= \frac{r}{2}(AB + BC + CA)
\end{aligned}$$

so

$$r = \frac{2[ABC]}{AB + BC + CA}$$

as desired.

47. Freya flips a coin 10 times. Compute the probability that she never flips a head immediately after flipping a tail.

Answer: $\mathbf{\frac{11}{1024}}$

Solution: To satisfy the problem statement, all tails must come after all heads. There can be anywhere from 0 to 10 heads before the first tail, giving 11 desired possibilities out of $2^{10} = 1024$ total possibilities for a probability of $\boxed{\dfrac{11}{1024}}$.

48. How many ordered pairs of integers (x, y) satisfy $x^2 + y^2 = 2^7$?

Answer: 4

Solution: Note that any perfect square can only leave a remainder 1 or 0 modulo 4. Since 2^7 is 0 modulo 4, both x^2 and y^2 must be 0 modulo 4, so x and y must both be even. Thus, we can let $x = 2x_1$ and $y = 2y_1$, where x_1 and y_1 are both integers. We then have $(2x_1)^2 + (2y_1)^2 = 2^7$, or $x_1^2 + y_1^2 = 2^5$. We can repeat this process again since both x_1 and y_1 must be even. We ultimately arrive at $x_3^2 + y_3^2 = 2$. Thus both x_3 and y_3 must each be either 1 or -1. In that case, x and y must each be either 8 or -8, and thus there are $\boxed{4}$ solutions.

49. A cube is painted black on all faces then cut into 1000 smaller congruent cubes. One of these smaller cubes is selected at random and rolled like a die. What is the probability that the top face is black?

Answer: $\mathbf{\frac{1}{10}}$ **OR 0.1**

Solution: Consider the faces among all 1000 of the smaller cubes. Each has equal probability of being the top face. There are $6 \cdot 10^2 = 600$ black faces and $6 \cdot 10^3 = 6000$ total faces, giving a probability of $\boxed{\dfrac{1}{10}}$ of the top face being black.

50. Given that
$$\frac{2^{2018}+1}{2^{2019}} = x$$
and
$$x^2 - x = -\frac{1}{4} + c,$$
compute the value of c.

Answer: $\frac{1}{2^{4038}}$

Solution: We can write $c = x^2 - x + \frac{1}{4} = \left(x - \frac{1}{2}\right)^2$, and observe that $x - \frac{1}{2} = \frac{1}{2^{2019}}$. Thus, $c = \boxed{\frac{1}{2^{4038}}}$.